Dynamic Patterns of Trade Imbalances
with Recursive Preference

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Abstract

Based on the recursive preference approach, the dynamic and global properties of the two-country open economy are examined when there exist one good used either for consumption or investment, and two inputs of labor and capital, with capital being freely and costlessly traded internationally. First the world's consumption is shown to increase with the increase in world's capital. Second, employing Devereux and Shi (1991)'s model, the home country which is more patient than the foreign country is shown to consume less than foreign country in the growing world economy, while in the decreasing world economy either, the home consumption remains less than that of the foreign country, or if the home consumption is larger than that of the foreign country initially then this difference is reversed after a while and reversed difference remains thereafter. Third, assuming further the Cobb-Douglas type production function, and assuming that the home country has technological superiority, the home country is shown to export good and to remain debtor throughout transitional period in the growing world economy, while in the decreasing world economy the possibility of different trade patterns arises. The rapid economic growth of some Asian countries such as Japan and Korea led by exports of good as engine of growth seems to be explained by our recursive preference model.

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I. Introduction

This paper tries to show the global properties of the two country open economy where the representative consumer maximizes not additive but recursive utility overtime with perfect foresight, under free capital mobility between two countries.

First the global stability of the world growing economy is derived. (Theorem 1). Next the consumption of the world is shown to increase globally with the increase in the world's capital (Theorem 2). Third, by restricting our model to Devereux and Shi (1991)'s simplified case, the consumption of the home country which is more patient than foreign county is shown to be less than that of the foreign country in the growing world economy (Theorem 3), while in the decreasing world economy the possibility of the reversed ranking is shown (Theorem 4). Next, assuming the Cobb-Douglas type production function, the home country is shown to export good and remain debtor throughout transitional periods in the growing world economy, while in the decreasing world economy either the same trade patterns and asset-debt position prevail from the beginning, or do so after a certain period. (Theorem 5).

The local stability of many heterogeneous agents with recursive preferences has been analyzed by Epstein (1987a), and its global stability also by Epstein (1987b). Devereux and Shi (1991) analyzed both trade imbalances and asset-debt position overtime near steady state based on the local stability of the two-country open economy model to whose works we owe a lot, although our emphasis is on not local but global properties of the economy.

One exception is Palivos, Wong and Zhang (1997) who obtained the global stability of the balanced growth path and its characteristics.

As is well known, quite a few authors have contributed to establish and elaborate the concept and the significance of recursive utility. Here we just mention a few of them. Uzawa (1968) first established this concept. Then Epstein (1983), (1987a) further extended. Obstfeld (1981) explained the significance of the recursive preference by developing models of exchange-rate and current account determination of a small open economy, and Epstein and Hynes (1983) by making several applications for macro economic topics respectively. Becker, Boyd and Sung (1989) provided the existence of the optimal capital accumulation paths of the recursive preference model. Obstfeld (1990) explained the significance of the recursive preference very heuristically and showed the global stability of the closed model.

Relating to our present works, Becker (1980) derived Ramsey's conjecture that in the long run steady state the income distribution is determined by the lowest discount rate. Buiter (1981) analyzed the asset-debt positions of a two country overlapping generation model. Lipton and Sachs (1983) analyzed the saving and capital accumulation of a two-good and two country model with time-additive preference employing simulation approach. Ikeda and Ono (1992) analyzed the dynamic patterns of trade imbalances within one commodity and multi country framework caused by a difference in discount rates.

In the next section, the basic framework of our model is introduced and the global stability of two country's open economy is derived. Basically we follow the framework of Epstein (1987a).

II. Basic Framework

II. 1 Social Planner's Optimum in Open Economy

First we investigate the global stability of the two country open economy. Following Devereux and Shi (1991), we assume there is one consumer in each country who supplies one unit of labor. First we analyze the case of social planner's optimum. Let $C=(C_1,C_2)$ be the streams of the consumption of the home country (1) and the foreign country (2) with $C_i=(c_i(t))_{t=0}^{\infty}$, $i=1,2$ and c_i is the per capita consumption of country *i* at time *t*. Let $k=k_1+k_2$ be the capital of the world where k_i being the capital of country *i*, $i=1,2$. Capital is freely and costlessly traded between two countries implying the marginal products of capital of both countries are equalized. Since the efficient production implies $f'_1(k_1)=f'_2(k_2)$, i.e., the equality of both countries' marginal products of capital given $k=k_1+k_2$, we can express $f(k)=f_1(k_1)+f_2(k_2)$ with $f'(k)=f'_1(k_1)=f'_2(k_2)$ where $f_i(k_i)$ being the production function of country *i*, $i=1,2$. f_i satisfies the Inada condition. $f'(k)=f_i'(k_i)$, $i=1,2$ implies that the capital of each country moves in the same direction, i.e., $\dot{k}_1>0 \Leftrightarrow \dot{k}_2>0 \Leftrightarrow \dot{k}>0$. The social planner tries to maximize the following utility;

$$\sum_i \alpha_i(0)U_i(C_i)=\sum_i \alpha_i(0)\int_0^{\infty} v_i(c_i(t))e^{-\int_0^t u_i(c_i(\tau))d\tau}dt \tag{1}$$

subject to the law of motion of capital

$$\dot{k}=f(k)-\sum_i c_i \tag{2}$$

and

$$\dot{\alpha}_i=-\alpha_i(t)u_i(c_i(t)) \tag{3}$$

where $\alpha_i(t)=\alpha_i(0)e^{-\int_0^t u_i(c_i(\tau))d\tau}$, $i=1,2$ and $\alpha_i(0)$ being the weight of the utility of the country *i*, $U_i(C_i)=\int_0^{\infty} v_i e^{-\int u_i d\tau}dt$, where $v_i(c_i)<0$ and $u_i(c_i)>0$ being the instantaneous felicity functions of country i, $i=1,2$. Recursive preferences are expressed by the endogenously determined intertemporal substitution rate of consumption $\int_0^t u_i(c_i)d\tau$. v_i and u_i are assumed to satisfy the regularity conditions for the existence of the optimal path (C_1, C_2); $v_i'(c)>0$, $\log(-v_i)$ being convex, $-\infty < \inf_{c_i>0} v_i(c) \le \sup_{c_i>0} v_i(c_i) < 0$, $u'_i(c_i)>0$, $u''_i(c_i)<0$ and $\inf_{c_i>0} u_i(c_i)>0$, $i=1,2$. (See Epstein (1987a), Lemma 1.) Furthermore $u_i(c_i)\to+\infty$ as $c_i \to +\infty$, $\lim_{c_i\to\infty} u'_i(c_i)=0$ and $\lim_{c_i\to\infty} v'_i(c_i)=0$ are assumed. Let H be the Hamiltonian

$$H=\sum_i \alpha_i v_i(c_i)+\lambda(f(k)-\sum_i c_i)-\sum_i \phi_i\alpha_i u_i(c_i) \tag{4}$$

and we obtain the first order conditions,

$$\dot{\lambda}=-\lambda f'(k), \tag{5}$$

$$\dot{\phi}_i=-v_i+\phi_i u_i\ , i=1,2 \tag{6}$$

$$\alpha_i v_i'(c_i)-\lambda-\phi_i\alpha_i u_i'(c_i)=0,\ i=1,2 \tag{7}$$

and the transversality conditions

$$\lambda k \to 0 \quad \text{as} \quad t\to\infty \tag{8}$$

and

$$\phi_i\alpha_i \to 0 \quad \text{as} \quad t\to\infty\ ,\ i=1,2. \tag{9}$$

Here by letting $\mu_i=\alpha_i/\lambda$, (3) and (5) are rewritten as

$$\dot{\mu}_i=\mu_i(f'(k)-u_i(c_i)), i=1,2 \tag{10}$$

and (7) as

$$\mu_i(v_i'-\phi_i u_i')=1,\ i=1,2. \tag{11}$$

By differentiating (11) with help of (6) and (10), we obtain

$$\dot{c}_i=-(v_i'-\phi_i u_i')(v_i''-\phi_i u_i'')^{-1}(f'-\rho_i) \tag{12}$$

where

$$\rho_i=(u_i v_i'-v_i u_i')(v_i'-\phi_i u_i')^{-1}>0. \tag{13}$$

It is known that equation (6) implies that ϕ_i is expressed as

$$\phi_i(t)=\int_t^\infty v_i(c_i(\tau))e^{-\int_t^\tau u_i(c_i(s))ds}d\tau<0,\ i=1,2 \tag{14}$$

which is the utility of the *i* country's consumer starting from initial time $t\ge 0$.

II. 2 Equivalence between Competitive Equilibrium and Social Planner's Optimum

Here in II. 2 we define competitive equilibrium in the open economy and first how it implies the social planner's optimum. In the competitive equilibrium, the consumer of country i tries to maximize

$$U_i(C_i) = \int_0^\infty v_i(c_i(t)) e^{-\int_0^t u_i(c_i(s))ds} dt, \quad i = 1,2 \tag{15}$$

subject to (3) and the budget constraint

$$\dot{m}_i = rm_i + w_i - c_i, \quad i = 1,2 \tag{16}$$

where $m_i > 0$ is the non-human wealth (abbreviated wealth) held by consumer i which is an equity claim on capital. Equities are traded internationally so that their interest rate is equal to rental price of capital, r by arbitrage conditions. From the profit maximization of firm

$$r = f_i'(k_i) = f'(k) \ , i = 1,2 \tag{17}$$

and

$$w_i = f_i(k_i) - k_i f_i'(k_i) \ , i = 1,2 \tag{18}$$

hold where w_i is the wage rate of country i. (17) implies that the capital of each country moves in the same direction, i.e., $\dot{k}_1 > 0 \Leftrightarrow \dot{k}_2 > 0$. Here

$$\sum_i m_i = \sum_i k_i = k \tag{19}$$

holds by definition. The utility maximization is solved by forming the Hamiltonian;

$$H_i = \alpha_i v_i(c_i) + \tilde{\lambda}_i(rm_i + w_i - c_i) - \tilde{\phi}_i \alpha_i u_i(c_i) \ , i = 1,2 \tag{20}$$

and by obtaining the first order conditions;

$$\alpha_i v_i'(c_i) - \tilde{\lambda}_i - \tilde{\phi}_i \alpha_i u_i'(c_i) = 0 \ , i = 1,2 \tag{21}$$

$$\dot{\tilde{\lambda}}_i = -\tilde{\lambda}_i r \ , i = 1,2 \tag{22}$$

$$\dot{\tilde{\phi}}_i = -v_i + \tilde{\phi}_i u_i \ , i = 1,2 \tag{23}$$

and the transversality conditions

$$\tilde{\lambda}_i m_i \to 0 \ \text{ as } \ t \to \infty \tag{24}$$

and

$$\tilde{\phi}_i \alpha_i \to 0 \ \text{ as } \ t \to \infty \ , i = 1,2 . \tag{25}$$

Here we note $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ to be proportionate from (22) and hence $\tilde{\lambda}_1 = \beta \tilde{\lambda}_2, \text{where } \beta > 0$ being constant. Let $\tilde{\lambda}_2 = \lambda$, $\tilde{\alpha}_1 = \alpha_1 / \beta$, $\tilde{\alpha}_2 = \alpha_2$, and $\tilde{\phi}_i = \phi_i \quad , i = 1,2$. Then (21) is rewritten as $\tilde{\alpha}_1 v_1' - \lambda - \phi_1 \tilde{\alpha}_1 u_1' = 0$ and $\tilde{\alpha}_2 v_2' - \lambda - \phi_2 \tilde{\alpha}_2 u_2' = 0$. Here by replacing α_1 and α_2 of the social planner's optimum by $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, we observe that the first order conditions (21), (22) and (23), the transversality conditions (24) and (25) of the competitive equilibrium satisfy those of the social planner's optimum (i.e., (5), (6) and (7), and the transversality conditions (8) and (9)). Here the budget constraint of the competitive equilibrium (16) implies the law of motion of capital (2) in view of

$$\sum_i (rm_i + w_i) = \sum_i (rk_i + w_i) = \sum_i f_i(k_i) = f(k).$$

In short, we observe that the competitive equilibrium implies the social planner's optimum. Conversely by letting $\phi_i = \tilde{\phi}_i, i = 1,2$, $\alpha_1 = \beta\tilde{\alpha}_1$, $\alpha_2 = \tilde{\alpha}_2$ and hence $\lambda = \lambda_1 / \beta = \lambda_2$, we observe that the social planner`s optimum implies the competitive equilibrium.

II. 3 Existence and Uniqueness of Stationary State of Competitive Equilibrium

The stationary state of the competitive equilibrium is obtained by letting $\dot{k} = 0, \dot{\mu}_i = 0, \dot{\phi}_i = 0, \dot{c}_i = 0$ and $\dot{m} = 0$. Hence we obtain at the stationary state *E*,

$$f(k) = \sum_i c_i = c, \tag{26}$$

$$u_i(c_i) = f'(k) = \rho_i, \ i = 1,2 \tag{27}$$

$$\phi_i = v_i(c_i)/u_i(c_i), \ i = 1,2 \tag{28}$$

and

$$m_i = (c_i - w_i)/r, \ i = 1,2 \tag{29}$$

Fig. 1

From (27), we observe $c_i = c_i(k)$ with $c_i'(k) < 0, \ i = 1,2$ and then from (26), the existence and the uniqueness of the stationary state *E* is immediate. We denote $c_i = \bar{c}_i, \phi_i = \bar{\phi}_i, m_i = \bar{m}_i, i = 1,2$ and $k = \bar{k}$ to be their respective values at the stationary state *E*. (Bar - sign is attached to denote the value at the stationary state *E*.)

II. 4 Local Representation of Competitive Equilibrium

Next, to show that the stationary state is locally a saddle point, we linearlize the above differential equations (12), (6), (2) and (16) of competitive equilibrium around the stationary state *E*, and obtain

$$\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{k} \\ \dot{m}_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -a_1 & 0 & -b_1 & 0 \\ 0 & 0 & 0 & -a_2 & -b_2 & 0 \\ -e_1 & 0 & \bar{r} & 0 & 0 & 0 \\ 0 & -e_2 & 0 & \bar{r} & 0 & 0 \\ -1 & -1 & 0 & 0 & \bar{r} & 0 \\ -1 & 0 & 0 & 0 & F_1 f'' & \bar{r} \end{pmatrix} \begin{pmatrix} c_1 - \bar{c}_1 \\ c_2 - \bar{c}_2 \\ \phi_1 - \bar{\phi}_1 \\ \phi_2 - \bar{\phi}_2 \\ k - \bar{k} \\ m_1 - \bar{m}_1 \end{pmatrix} \tag{30}$$

where $a_i = -(v_i'' - \phi_i u_i'')^{-1} \bar{r} u_i' > 0$, $b_i = -(v_i' - \phi_i u_i')(v_i'' - \phi_i u_i'')^{-1} f'' > 0$ and

$e_i = v_i' - \phi_i u_i' > 0$, $i = 1,2$ (a_i, b_i and e_i are evaluated at E), $\bar{u}_i = \bar{\rho}_i = \bar{f}' = f'(\bar{k})$ $= r$, $i = 1,2$ (See Appendix I.) and $F_1 = m_1 - k_1$ is the home country's net foreign asset holding.

Let A and B be respectively the coefficient matrix of (30) and its 5x5 upper and left submatrix. Then we obtain for

$$|A - \omega I| = (\bar{r} - \omega)(B - \omega I) \text{ and hence for } |B - \omega I| = 0 \tag{31}$$

two negative ω_1 and ω_2 such that

$$\omega_i = (-\bar{r} - \sqrt{\bar{r}^2 - 4\lambda_i})/2 < 0 \text{ , } i = 1,2 \tag{32}$$

where λ_1 and λ_2 are two negative solutions of

$$h(\lambda) = \lambda^2 + (a_1 e_1 + a_2 e_2 + b_1 + b_2)\lambda + a_1 a_2 e_1 e_2 + e_2 a_2 b_1 + e_1 a_1 b_2 = 0$$

with $\lambda_2 < \lambda_1 < 0$.(See Appendix I.)[1/] Then $\omega_2 < \omega_1 < 0$ follows. In short we can conclude that there exists locally a two dimensional manifold of the optimal path of (c_1, c_2, ϕ_1, ϕ_2, k) which converges monotonically to the stationary state as a saddle point. Here the local representation of the optimal path is given for $\lambda_i \neq -a_1 e_1$, $i = 1,2$, by

$$c_1 - \bar{c}_1 = -A_1 \frac{b_1(\bar{r} - \omega_1)}{\lambda_1 + a_1 e_1} e^{\omega_1 t} - A_2 \frac{b_1(\bar{r} - \omega_2)}{\lambda_2 + a_1 e_1} e^{\omega_2 t} \tag{33-1}$$

$$c_2 - \bar{c}_2 = -A_1 \frac{b_2(\bar{r} - \omega_1)}{\lambda_1 + a_2 e_2} e^{\omega_1 t} - A_2 \frac{b_2(\bar{r} - \omega_2)}{\lambda_2 + a_2 e_2} e^{\omega_2 t} \tag{33-2}$$

$$\phi_1 - \bar{\phi}_1 = -A_1 \frac{e_1 b_1}{\lambda_1 + a_1 e_1} e^{\omega_1 t} - A_2 \frac{e_1 b_1}{\lambda_2 + a_1 e_1} e^{\omega_2 t} \tag{33-3}$$

$$\phi_2 - \bar{\phi}_2 = -A_1 \frac{e_2 b_2}{\lambda_1 + a_2 e_2} e^{\omega_1 t} - A_2 \frac{e_2 b_2}{\lambda_2 + a_2 e_2} e^{\omega_2 t} \tag{33-4}$$

$$k - \bar{k} = A_1 e^{\omega_1 t} + A_2 e^{\omega_2 t} \tag{33-5}$$

and

$$m_1 - \bar{m}_1 = -A_1 \left(\frac{b_1}{\lambda_1 + a_1 e_1} + \frac{\bar{F}_1 f''(\bar{k})}{\bar{r} - \omega_1} \right) e^{\omega_1 t} - A_2 \left(\frac{b_1}{\lambda_2 + a_1 e_1} + \frac{\bar{F}_1 f''(\bar{k})}{\bar{r} - \omega_2} \right) e^{\omega_2 t} . \tag{33-6}$$

For $\lambda_1 = -a_1 e_1$ and $\lambda_2 = -a_1 e_1 - b_1 - b_2$

$$
\begin{cases}
c_1-\bar{c}_1 = A_1 e^{\omega_1 t} + A_2 \dfrac{b_1}{b_1+b_2}(\bar{r}-\omega_2)e^{\omega_2 t} & (34\text{-}1)\\
c_2-\bar{c}_2 = -A_1 e^{\omega_1 t} + A_2 \dfrac{b_2}{b_1+b_2}(\bar{r}-\omega_2)e^{\omega_2 t} & (34\text{-}2)\\
\phi_1-\bar{\phi}_1 = -A_1 \dfrac{\omega_1}{a_1} e^{\omega_1 t} + A_2 \dfrac{e_1 b_1}{b_1+b_2} e^{\omega_2 t} & (34\text{-}3)\\
\phi_2-\bar{\phi}_2 = A_1 \dfrac{\omega_1}{a_2} e^{\omega_{12} t} + A_2 \dfrac{e_2 b_2}{b_1+b_2} e^{\omega_2 t} & (34\text{-}4)\\
k-\bar{k} = A_2 e^{\omega_2 t} & (34\text{-}5)\\
\text{and} & \\
m_1-\bar{m}_1 = \dfrac{A_1}{\bar{r}-\omega_1} e^{\omega_1 t} + A_2\left(\dfrac{b_1}{b_1+b_2} - \dfrac{\bar{F}_1 f''}{\bar{r}-\omega_2}\right)e^{\omega_2 t} & (34\text{-}6)
\end{cases}
$$

where A_1 and A_2 are determined by the initial values of k and net foreign asset holding F;

$$
A_1 = \begin{cases}
\dfrac{m_{10}-\bar{m}_1+\{b_2/(\lambda_2+a_1e_1)+\bar{F}_1 f''(\bar{k})/(\bar{r}-\omega_2)\}(k_0-\bar{k})}{-b_1/(\lambda_1+a_1e_1)+b_1/(\lambda_2+a_1e_1)-\bar{F}_1 f''(\bar{k})\{(\bar{r}-\omega_1)^{-1}-(\bar{r}-\omega_2)^{-1}\}} & (35\text{-}1)\\
\text{for } \lambda_i \neq -a_1e_1,\ i=1,2 & \\
\left\{m_{10}-\bar{m}_1-(k_0-\bar{k})\left(\dfrac{b_1}{b_1+b_2}-\dfrac{\bar{F}_1 f''(\bar{k})}{\bar{r}-\omega_2}\right)\right\}(\bar{r}-\omega_1) & \\
\text{for } \lambda_1=-a_1e_1,\ \lambda_2=-a_1e_1-b_1-b_2 & (35\text{-}2)
\end{cases}
$$

$$
A_2 = \begin{cases}
\dfrac{-\{b_1/(\lambda_1+a_1e_1)+\bar{F}_1 f''(\bar{k})/(\bar{r}-\omega_1)\}(k_0-\bar{k})-m_{10}+\bar{m}_1}{-b_1/(\lambda_1+a_1e_1)+b_1/(\lambda_2+a_1e_1)-\bar{F}_1 f''(\bar{k})\{(\bar{r}-\omega_1)^{-1}-(\bar{r}-\omega_2)^{-1}\}} & (36\text{-}1)\\
\text{for } \lambda_i \neq -a_1e_1,\ i=1,2 & \\
k_0-\bar{k} \text{ for } \lambda_1=-a_1e_1,\ \lambda_2=-a_1e_1-b_1-b_2 & (36\text{-}2)
\end{cases}
$$

where k_0 and m_{10} are respectively the initial values of $k=k(t)$ and $m_1=m_1(t)$, i.e., $k_0=k(0)$ and $m_{10}=m_1(0)$.

First we note that $c=c_1+c_2$ is locally an increasing function of k. In fact for $\lambda_i \neq -a_1e_1$, $i=1,2$, from $c-\bar{c}=A_1(\bar{r}-\omega_1)e^{\omega_1 t}+A_2(\bar{r}-\omega_2)e^{\omega_2 t}$, (33-5) and $\omega_2<\omega_1<0$, we obtain $(c-\bar{c})/(k-\bar{k}) \to (\bar{r}-\omega_1)>0$ as $t\to\infty$. Here $(\lambda_i+a_1e_1)(\lambda_i+a_2e_2)+b_1(\lambda_i+a_2e_2)+b_2(\lambda_i+a_1e_1)=h(\lambda_i)=0$, $i=1,2$ is employed. Similarly for $\lambda_1=-a_1e_1$, $\lambda_2=-a_1e_1-b_1-b_2$, from $c-\bar{c}=A_2(\bar{r}-\omega_2)e^{\omega_2 t}$ and (34-5), we obtain $(c-\bar{c})/(k-\bar{k})=(\bar{r}-\omega_2)>0$. Next we generalize this local property of c as an increasing function of k into global one.

As for the local representation of c_i, $i=1,2$ as a function of k, for $\lambda_i \neq -a_1e_1$, $i=1,2$, we note from (33-1), (33-2) and (33-5), $(c_1-\bar{c}_1)/(k-\bar{k})\to -b_1(\bar{r}-\omega_1)/(\lambda_1+a_1e_1)$ and $(c_2-\bar{c}_2)/(k-\bar{k})\to -b_2(\bar{r}-\omega_1)/(\lambda_1+a_2e_2)$ as $t\to\infty$. Since $(\lambda_i+a_1e_1)(\lambda_i+a_2e_2)$ <0 holds for $\lambda_i\neq -a_1e_1$, (See Appendix I), $c_1'(k)$ and $c_2'(k)$ are seen to be of opposite sign. Similarly for $\lambda_1=-a_1e_1$ and $\lambda_2=-a_2e_2-b_1-b_2$, recalling $e^{(\omega_2-\omega_1)t}\to 0$ as $t\to\infty$, we note $(c_1-\bar{c}_1)/(k-\bar{k})\to +\infty$ (resp. $-\infty$) if $A_1/A_2>0$ (resp. <0) and $(c_2-\bar{c}_2)/(k-\bar{k})\to -\infty$ (resp. $+\infty$) if $A_1/A_2>0$ (resp. <0), and hence $c_1'(k)$ and $c_2'(k)$ are of opposite sign.

III. Global Stability of Competitive Equilibrium

III. 1 Global Properties of the Consumption Path *C*

Next, we show the global stability. Here we first show that c_i and ϕ_i, $i=1,2$ are continuously differentiable functions of *k*.

Let $X=(t,c_1,c_2,\phi_1,\phi_2,k)$ be the solution path given by the system of ordinary differential equations (2), (5) and (11);

$$\dot{k}=f(k)-(c_1+c_2) \tag{2}$$

$$\dot{\phi}_i=-v_i(c_i)+\phi_i u_i(c_i),\quad i=1,2 \tag{5}$$

$$\dot{c}_i=-(v_i'-\phi_i u_i')(v_i''-\phi_i u_i'')^{-1}(f'-\rho_i(c_i,\phi_i)),\quad i=1,2. \tag{12}$$

Then employing the fundamental theorem of ordinary differential equations (see, e.g., Hurewicz (1958, Theorems 8, 9 and 11, pp.29-32)) [2/] we express $\dot{c}_i/\dot{k}=dc_i/dk=$

$-(v_i'-\phi_i u_i')(v_i''-\phi_i u_i'')^{-1}(f'-\rho_i)/(f(k)-c_1-c_2)$, and $\dot{\phi}_i/\dot{k}=d\phi_i/dk=$ $(-v_i+\phi_i u_i)$ $/(f(k)-c_1-c_2)$, $i=1,2$ as functions of *k*, c_i and ϕ_i, $i=1,2$ and by replacing the role of *t* with *k*, we obtain $c_i=c_i(k,\ Y_0)$ and $\phi_i=\phi_i(k,\ Y_0)$, $i=1,2$ to be continuously differentiable in $(k,\ Y_0)$ for *k*, c_1 and c_2 such that $\dot{k}=f(k)-(c_1+c_2)\neq 0$ where Y_0 is the initial value of $Y=(c_1,c_2,\phi_1,\phi_2)$. Here we note both $c_i(k,\ Y_0)$ and $\phi_i(k,\ Y_0)$, $i=1,2$ to be uniquely expressed for each $\dot{k}=f(k)-(c_1+c_2)>0$ and $\dot{k}=f(k)-(c_1+c_2)>0$ by construction. Then it is possible for both $c_i(k,\ Y_0)$ and $\phi_i(k,\ Y_0)$ to have two values for a given *k*. Bearing this in mind, however, we retain the same expression, $c_i(k,\ Y_0)$ and $\phi_i(k,\ Y_0)$, for simplicity. It is immediate to show the functions c_i and ϕ_i are continuously differentiable in (k,Y_0) for $\dot{k}=f(k)-(c_1+c_2)=0$ as well except at $k=\bar{k}$ and hence the functions c_i and ϕ_i are continuously differentiable in (k, Y_0) for all $k > 0$ except $k = \bar{k}$. [3/]

Although c_i and ϕ_i are functions of k as well as t so that these are expressed as $c_i = \tilde{c}_i(t,\ Y_0) = c_i(k,\ Y_0)$ and $\phi_i = \tilde{\phi}_i(t,\ Y_0) = \phi_i(k,\ Y_0)$, $i = 1,2$, hence forth we abuse notation to express as $c_i = c_i(k)$ and $\phi_i = \phi_i(t) = \phi_i(k)$, $i = 1,2$, whenever there exists no danger of confusion.

Hence we obtain that the optimal path $Y = (c_1(k), c_2(k), \phi_1(k), \phi_2(k))$ converges at least locally to $\bar{Y} = (c_1(\bar{k}), c_2(\bar{k}), \phi_1(\bar{k}), \phi_2(\bar{k})) = (\bar{c}_1, \bar{c}_2, \bar{\phi}_1, \bar{\phi}_2)$.

Next we show the global property of the optimal path *Y*. To show this, we employ the following lemmas;

Lemma 1

c_1, c_2, ϕ_1, ϕ_2 and *k* are bounded.

Proof

ϕ_i, $i = 1,2$ is seen to be bounded from (14);

$$|\phi_i(t)| \leq \int_t^\infty |v_i(c_i(t)| e^{-\int_t^\tau u_i(t)d\tau} dt \leq -\sup_{c_i>0} v_i(c_i) \int_t^\infty e^{-(\inf_{c_i\geq 0} u_1)(\tau - t)} dt$$

$$= -\sup_{c_i>0} v_i(c_i) / \inf_{c_i\geq 0} u_i(c_i) \equiv \Phi < +\infty .$$

Next we show c_i, $i = 1,2$ to be bounded.

Suppose C is not bounded. Then either C_1 or C_2 is unbounded. We assume without loss of generality, C_1 is unbounded, i.e., $c_1 \to \infty$ as $t \to \infty$. Then $k \to \infty$ as $t \to \infty$ from (2), ϕ_i is bounded, and $\mu_1 \to +\infty$ from (11) and $\lim_{c_i\to\infty} v_i'(c_i) = \lim_{c_i\to\infty} u_i'(c_i) = 0, i = 1,2$. From (10), μ_1 cannot be infinite since $f'(k) - u_1(c_1) < 0$ for sufficiently large k and c_1. ($f'(k) \to 0$ as $k \to \infty$ from the Inada conditions.) This contradiction shows both C_1 and C_2 , and hence C to be bounded.

Now suppose *k* is not bounded. Then from (2) there exist $\varepsilon_0 > 0$ and $t_1 > 0$ such that

$$f(k) > 2\sup C + \varepsilon_0 \qquad \text{for any} \qquad t > t_1 .$$

Then we can consider the suboptimal path C' such that $C' = C$ for $t \leq t_1$ and

$C'=(C_1',C_2')$, $C_1'=\sup C+\frac{1}{2}\varepsilon_0$, $C'_2=\sup C+\frac{1}{2}\varepsilon_0$ for $t>t_1$. This suboptimal path C' causes higher utility than C, contradicting the optimality of C. Hence k must also be bounded. ■

Lemma 2 (Poincaré-Bendixon Theorem) [4/]
For the two dimensional autonomous differential equation system, the path (trajectory) must become unbounded or converge to a limit cycle or to a point.
Recalling that $(c_1,k)=(c_1(k),k)$ converges locally to $(\bar{c}_1,\bar{k})$ and hence from Lemma 1 and 2, we observe that it does also globally. We can employ the same arguments for c_2, ϕ_1 and ϕ_2. Hence we observe the following theorem;

Theorem 1
There exists a solution path $Y=(c_1,c_2,\phi_1,\phi_2)=(c_1(k,Y_0),c_2(k,Y_0),\phi_1(k,Y_0),\phi_2(k,Y_0))$ which converges to $\bar{Y}=(\bar{c}_1,\bar{c}_2,\bar{\phi}_1,\bar{\phi}_2)$ as $k\to\bar{k}$ for a given k_0, under an appropriate choice of Y_0.

Now we analyze the global properties of the optimal path of $c=c(k)$ employing the system of ordinary equations (38).

Fig. 2a **Fig. 2b**

Although the sign of $\xi'(k)$ at $k=\bar{k}$ seems not to be definite, in case of $\xi'(k)\le 0$ (Fig. 2a), the economy is globally stable in the sense it converges monotonically to a stationary point, but in case of $\xi'(k)>0$ (Fig. 2b), the economy converges to a limit cycle or to a point (as a spiral node) as shown by Lemmas 1 and 2. However as shown above in (33) and (34), the stationary point is locally a stable saddle point. Furthermore from the above arguments, in particular we obtain for the world's consumption c;

Theorem 2
For the open economy, the optimal path of world consumption c and world per capita capital k is globally stable so that

$c=c(k)$ with $k\to\bar{k}$ as $t\to\infty$ monotonically and $c'(k)>0$.

III. 2 Characteristics of Consumption Path
Here we investigate the characteristics of the consumption path of both countries. The

felicity function v_i,i=1,2 is identically equal to -1, i.e.,

$$v_1(c_1) = v_2(c_2) = -1 .$$

The felicity function u_i reflects that the home country is more impatient than the foreign country, i.e.,

$$u_1(c) > u_2(c) \quad \text{for any } c.$$

We specify

$$u_1(c_1) = \log(c_1 + \alpha) + \gamma,\ \alpha \text{ and } \gamma > 0$$

and

$$u_2(c_2) = \log(c_2 + \alpha)$$

following Devereux and Shi (1991). Here u_i satisfies the regularity conditions

mentioned before. ($v = -1$ also satisfies their conditions.) This specification implies that there is no distributional effects (i.e., marginal propensity to save out of wealth is the same), yet time preference rates of consumption are different between two countries. For $\gamma > 0$, $u_1(\bar{c}_1) = u_2(\bar{c}_2) = \bar{\rho} = f(\bar{k})$ implies $\bar{c}_1 < \bar{c}_2$. Furthermore, under this specification $a_1 e_1 = a_2 e_2$, $\lambda_1 = -a_1 e_1$ and $\lambda_2 = -a_1 e_1 - b_1 - b_2$ hold. Hence we show $c_1 < c_2$ holds always.

Fig. 3a **Fig. 3b**

a for the local properties of the slopes of c_i, $i = 1,2$ curves, we have seen already that

$$c'_1(k) \to -\infty \text{ (resp. } +\infty \text{) and } c'_2(k) \to +\infty \text{ (resp. } -\infty \text{) as } k \leftarrow \bar{k}$$

if and only if $A_1 / A_2 < 0$ (resp. > 0). Fig. 3a (resp. Fig. 3b) corresponds to $A_1 / A_2 < 0$ (resp. > 0). First we are concerned with the case of growing world economy, i.e., $k < \bar{k}$.

First we show that for $k < \bar{k}$, $\operatorname{sgn} c_1'(k)$ (resp. $\operatorname{sgn} c_2'(k)$) changes only once while $\operatorname{sgn} c_2'(k)$ (resp. $\operatorname{sgn} c_1'(k)$) > 0 holds always for $A_1 / A_2 < 0$ (resp. $A_1 / A_2 > 0$).

Since $c'(k) > 0$ holds globally, $c(k) \to 0$ as $k \to 0$ and $c_1 \le c$ and must hold always, we observe both c_1 and c_2 must decrease to zero as $k \to 0$ for both $A_1 / A_2 < 0$ and $A_1 / A_2 > 0$ cases. Recalling that (6) is reduced to

$$\dot{\phi}_i = 1 + \phi_i u_i(c_i),\ \ i = 1,2$$

Fig. 4

and $u_i(c_i) = u_i(c_i(k))$, $i = 1,2$, we observe

$$\operatorname{sgn} d\phi_i / dk \Big|_{\dot{\phi}_i = 0} = \operatorname{sgn}\left(-\frac{\phi_i u_i' \cdot c_i'(k)}{u_i} \right) = \operatorname{sgn} c_i'(k) .$$

Hence $\operatorname{sgn} d\phi_i / dk\big|_{\dot{\phi}_i=0} > 0$ if and only if $c_i'(k) > 0$. By definition of $\dot{\phi}_i = 1 + \phi_i u_i(c_i)$, $\dot{\phi}_i < 0$ (resp. > 0) if and only if $\phi_i = \phi_i(k)$ curve is below (above) $\dot{\phi}_i = 0$ curve as seen from Fig. 4. Then $\phi_i = \phi_i(k)$ curve passes through a_1 and a_2 where $\dot{\phi}_i = 0$ as shown in Fig. 4.

Hence we obtain

$$\phi_i'(k) > 0 \Leftrightarrow c_i'(k) > 0 \text{ , } i = 1,2$$

Since

$$\dot{c}_i = -(u_i'/u_i'')(f'(k) - \rho_i) \text{ , } i = 1,2 \tag{39}$$

holds from (12) and $\rho_i = -1/\phi_i$ from (13)

$$\rho_i'(k) > 0 \Leftrightarrow \phi_i'(k) > 0 \text{ , } i = 1,2 \tag{40}$$

where $\rho_i = \rho_i(k), \quad i = 1,2$, we obtain

$$c_i'(k) > 0 \Leftrightarrow f'(k) > \rho_i \text{ , } i = 1,2$$

from (39) and $\dot{k} > 0$ for $k < \bar{k}$.

Fig. 5

Now we are ready to show $\operatorname{sgn} c_1'(k)$ changes only once. Suppose for a contradiction, there exists a_1 and a_2 where $\phi_i'(k) = 0$ holds. Let a_1 be the point where $c_i'(k) > 0$ for $k < k^1$, $c_i'(k^1) = 0$, $c_i'(k) < 0$ for $k^1 < k < k^2$ as shown in Figs. 4 and 5. Then from (39) and (40),

$$f'(k) > \rho_i(k) \text{ for } k < k^1,$$

$$f'(k^1) = \rho_i(k^1),$$

$$f'(k) < \rho_i(k) \text{ for } k^1 < k < k^2,$$

$$f'(k^2) = \rho_i(k^2)$$

and

$$f'(k) > \rho_i(k) \text{ for } k^2 < k < k^3 \le \bar{k}.$$

However since $\phi_i'(k^2) = 0$ holds, so does $\rho_i'(k^2) = 0$ from (40), contradicting $\rho_i'(k^2) < 0$ as shown in Fig. 5. ($\rho_i'(k^2) < 0$ must hold for ρ_i curve to intersect with

$f'(k)$ curve.) This contradiction shows $f'(k) = \rho_i(k)$ holds at most only one *k*. Figs. 3a and 3b show respectively the cases where $\operatorname{sgn} c_1'(k)$ and $\operatorname{sgn} c_2'(k)$ change only once.

From this argument, we also obtain that c_2 curve in Fig. 3a and c_1 curve in Fig. 3b never changes its $\operatorname{sgn} c_2'(k)$ and $\operatorname{sgn} c_1'(k)$ respectively for $k < \bar{k}$. In fact, in Fig. 3a, for example, since $c_2'(k) > 0$ both as $k \to 0$ and $k \to \bar{k}$ holds, if $\operatorname{sgn} c_2'(k)$ ever changes, it changes at least twice, contradicting the change in $\operatorname{sgn} c_2'(k)$ to be at most just once.

Next we show c_1 curve and c_2 curve never meet except at $k = 0$, and hence $c_1 < c_2$ holds always for $0 < k \le \bar{k}$.

First we construct the straight line of (c_1, c_2) which satisfies $u_1(c_1) = u_2(c_2)$, or $\log(c_1 + \alpha) + \gamma = \log(c_2 + \alpha)$ with α and $\gamma > 0$, which is equivalent to $c_2 + \alpha = \beta(c_1 + \alpha)$ with $\beta = e^{\gamma} > 1$, as is shown in Fig. 6.

Fig. 6

I. c_1 curve and c_2 curve never intersect.

We here first show c_1 curve and c_2 curve never intersect. We first investigate the case where $A_1 / A_2 < 0$ (Fig. 3a). Suppose, for a contradiction, c_1 curve and c_2

curve intersect at least twice at E_1 and E_2 as shown in Fig. 7a. In Fig. 6, the curve starting from *E* passing through E_2 and E_1, corresponds to the movement of c_1 and c_2 curves starting from $k = \bar{k}$ in Fig. 7a. As seen in Fig. 6, the curve EE_2E_1 is below the straight line $c_2 + \alpha = \beta(c_1 + \alpha)$, implying $u_1(c_1) > u_2(c_2)$ for $t > t^1$ where $k^1 = k(t^1)$ and $(c_1^1,\ c_2^1) = E_1$ with $c_i^1 = c_i(k^1)$, $i = 1,2$. In short at E_1 and thereafter (i.e., $t > t^1$) $u_1(c_1) > u_2(c_2)$ holds always and hence $\phi_1(t^1) > \phi_2(t^1)$ by construction. However as seen from Fig. 7a, at E_1 $dc_1/dk \ge dc_2/dk$ holds and hence from (39) and $\rho_i = -1/\phi_i$, $i = 1,2$, $\rho_1 \le \rho_2$ or $\phi_1(t^1) \le \phi_2(t^1)$ must hold, a contradiction. Here we note that the above arguments hold especially when E_1, happens to be the origin.

Hence for $A_1 / A_2 < 0$, c_1 curve and c_2 curve never intersect for $0 < k \le \bar{k}$.

Fig. 7b

Fig. 7b corresponds to the case of $A_1 / A_2 > 0$, assuming c_1 curve and c_2 curve intersect at least twice at E_1 and E_2, for a contradiction. Since $dc_1/dk \ge dc_2/dk$ holds at E_1 where $t = t^1$,

$$\phi_1(t^1) = -\int_{t^1}^{\infty} e^{-\int_{t^1}^{\tau} u_1 ds} d\tau \le -\int_{t^1}^{\infty} e^{-\int_{t^1}^{\tau} u_2 ds} d\tau = \phi_2(t^1) \tag{41}$$

follows from (39) and $\rho_i = -1/\phi_i$ again. On the other hand, since $c_2 \le c_1$ for $t^1 \le t \le t^2$ where $t = t^2$ holds at E_2, $u_1(c_1) \ge u_2(c_2)$ for $t^1 \le t \le t^2$, implying

$$-\int_{t^1}^{t^2} e^{-\int_{t^1}^{\tau} u_1 ds} d\tau > -\int_{t^1}^{t^2} e^{-\int_{t^1}^{\tau} u_2 ds} d\tau . \tag{42}$$

Now, since $dc_1 / dk \le dc_2 / dk$ holds at E_2 where $t = t^2$

$$\phi_1(t^2) = -\int_{t^2}^{\infty} e^{-\int_{t^2}^{\tau} u_1 ds} d\tau \ge -\int_{t^2}^{\infty} e^{-\int_{t^2}^{\tau} u_2 ds} d\tau = \phi_2(t^2) \tag{43}$$

must holds. Here recalling $e^{-\int_{t^1}^{t2} u_1 ds} < e^{-\int_{t^1}^{t2} u_2 ds}$ holds for $t^1 \le t \le t^2$, we obtain from (43)

$$-\int_{t_1}^{\infty} e^{\left(-\int_{t^1}^{t^2} u_1 ds - \int_{t^2}^{\tau} u_1 ds\right)} d\tau \ge \;] -\int_{t_1}^{\infty} e^{\left(-\int_{t^1}^{t^2} u_2 ds - \int_{t^2}^{\tau} u_2 ds\right)} d\tau ,$$

or

$$-\int_{t^1}^{\infty} e^{-\int_{t^2}^{\tau} u_1 ds} d\tau \ge -\int_{t^1}^{\infty} e^{-\int_{t^2}^{\tau} u_2 ds} d\tau . \tag{44}$$

By re-expressing (41) as

$$-\int_{t_1}^{t^2} e^{-\int_{t^1}^{\tau} u_1 ds} d\tau - \int_{t^2}^{\infty} e^{-\int_{t^2}^{\tau} u_1 ds} d\tau \le -\int_{t_1}^{t^2} e^{-\int_{t^1}^{\tau} u_2 ds} d\tau - \int_{t^2}^{\infty} e^{-\int_{t^2}^{\tau} u_2 ds} d\tau .$$

and from (44), we observe

$$-\int_{t^1}^{t^2} e^{-\int_{t^1}^{\tau} u_1 ds} d\tau < -\int_{t^1}^{t^2} e^{-\int_{t^1}^{\tau} u_2 ds} d\tau ,$$

contradicting (42). We here note that the above arguments hold especially when E_1, happens to be the origin. Hence this contradiction shows c_1 curve and c_2 curve never intersects for $0 < k \le \bar{k}$.

II. c_1 curve and c_2 curve never touch.

Fig. 8

Next we show c_1 curve and c_2 curve never touch. First we investigate the case where $A_1 / A_2 < 0$. For this case we can employ the similar arguments as I, with $t^1 = t^2$. Suppose, for a contradiction $c_1 = c_2$ at $E_1 = E_2$ and $c_1 < c_2$ but $u_1(c_1) > u_2(c_2)$ for *t* with $0 < t < +\infty$ as can be seen in Fig. 6 with $E_1 = E_2$. Then by construction

$$\phi_1(t_1) > \phi_2(t_1)$$

must hold. However at $E_1 = E_2$ as seen from Fig. 8 $dc_1 / dk = dc_2 / dk$ holds, and hence

$$\phi_1(t_1) = \phi_2(t_1)$$
from (39) and $\rho_i = -1/\phi_i$, $i = 1,2$, a contradiction. This shows c_1 curve and c_2 curve never touch as shown in Fig. 8.

Next we investigate the case where $A_1 / A_2 > 0$. Recalling the slope of $\dot{\phi}_i = 0$ is positive if and only if $c_i' > 0$, and from Fig. 4, we obtain the two curves $\phi_1(k)$ and

Fig. 9

$\phi_2(k)$ for case $A_1 / A_2 > 0$ as drawn in Fig. 9 assuming two curves meet at E_1 . ($\phi_1(k)$ curve is positively sloped from $c_1' > 0$ for $0 < k < \bar{k}$ and $c_2(k)$ curve changes its sign from positive to negative just once as k increases up to $\bar{k}$. Then employing Fig. 4, we can obtain the slopes of two curves as drawn in Fig. 9.) Hence $\phi_1(t_1) = \phi_2(t_1)$, i.e., (two curves ϕ_1 and ϕ_2 meet at E_1) is obtained from $dc_1 / dk = dc_2 / dk$ at E_1 as drawn in Fig. 8, (39), and $\rho_i = -1/\phi_i$, $i = 1,2$. Here the slope of ϕ_2 curve is higher than that of ϕ_1 curve, i.e.,
$$0 < d\phi_1 / dk < d\phi_2 / dk \quad \text{at } E_1$$
comes from $u_1 > u_2$, $\phi_1 = \phi_2$ and $\dot{\phi}_i = 1 + \phi_i u_i > 0$ at E_1. This implies at k' slightly smaller than k^1 such that $k' = k(t')$
$$\phi_1(t') > \phi_2(t')$$
must hold. However then from Fig. 8(Fig. 8 holds for both $A_1 / A_2 < 0$ and $A_1 / A_2 > 0$.), at k'
$$dc_2 / dk < dc_1 / dk$$
must follow, implying $0 < (c_2 + \alpha)(f' - \rho_2) < (c_1 + \alpha)(f' - \rho_1)$ from (39), and hence $(f' - \rho_2) < (f' - \rho_1)$ from $c_1 < c_2$ at k' as shown in Fig. 8. This further implies $\rho_1 < \rho_2$ and hence $\phi_1(t') < \phi_2(t')$ from $\rho_i = -1/\phi_i$, a contradiction. This contradiction shows c_1 curve and c_2 curve never touch for $A_1 / A_2 > 0$.
In short, we obtain

Theorem 3
Let $v_i(c_i) = -1$, $i = 1,2$, $u_1(c_1) = \log(c_1 + \alpha) + \gamma$, and $u_2(c_2) = \log(c_2 + \alpha) + \gamma$ where α and $\gamma > 0$. Let $k < \bar{k}$, i.e., the world economy be increasing.
Then
(1) the more patient home country's consumption c_1 is always less than that of the foreign country.
(2) If the consumption of both countries increases as the world's capital stock increases, then it continuous to increase after a while, but one country's consumption starts

decreasing while the other country's consumption keeps increasing.
(3) If one country's consumption decreases, while the other country's consumption increases, then this consumption pattern remains unchanged.

Next we show the corresponding results for the decreasing world economy, i.e., $\bar{k} < k$.

Theorem 4
Let the world economy be decreasing, i.e., $\bar{k} < k$.
Then
(1) the more patient home country's consumption c_1 is always less than that of the foreign county, or if the home country's consumption is larger than that of the foreign country, then the home country's consumption becomes less than that of the foreign country after a certain time.
(2) If the consumption of both countries decreases as the world capital stock decreases, then it countinues to decrease after a while but one country's consumption starts increasing while the other country's consumption keeps decreasing.
(3) If one country's consumption decreases, while the other country's consumption increases, then this consumption pattern remains unchanged.
Proof (See Appendix II)

The only difference from increasing world economy's case is the possibility of the change in the amounts of consumption between two countries although this change occurs only once. As shown later this possibility gives rises to the possibility of the change in trade pattern and asset-debt position.

Theorems 3 and 4 show the significance of global analysis in comparison with the local analysis. By restricting to the local analysis, we observe only the difference in the direction of two countries' consumption path. However by generalizing to the global analysis we observe that when the starting points is not close to the stationary state, this direction is the same initially, and eventually the change in the direction of only one country occurs toward the stationary state. Such a non monotonicity of one country's consumption while monotonicity of the other country's consumption with respect to world capital increase is shown to be observed only by global analysis.

Next we investigate the trade patterns and asset-debt positions.

III. 3 Trade Patterns and Asset-Debt Positions
Henceforth we specify the production functions to be of Cobb-Douglas type, i.e.,

$$\left.\begin{aligned} f_1(k_1) &= \theta k_1^{\xi}, \theta \ge 1 \text{ and } 0<\xi<1 \\ \text{and} \qquad f_2(k_2) &= k_2^{\xi} \end{aligned}\right\} \tag{45}$$

That is, the home country is assumed to be technologically at least as good as the foreign country, $\theta \ge 1$.

Next we show when $c_1 < c_2$ holds

$$ex_1 = f_1(k_1) - c_1 - \dot{k}_1 > 0, \tag{46}$$

i.e., the home country's export which is the home output $f_1(k_1)$ less the home consumption c_1 and the home investment $\dot{k}_1$ is always positive if the home consumption c_1 is less than that of the foreign country. Since $f_1'(k_1) = f_2'(k_2) = r$ holds always, we obtain $k_1/k_2 = \theta^{1/(1-\xi)} = \eta/(1-\eta)$ where $\eta = \theta^{1/(1-\xi)}/(1+\theta^{1/(1-\xi)})$ with $\eta'(\theta) > 0$ and $\eta = 1/2$ for $\theta = 1$. Then $(1-\eta)k_1 = \eta k_2$ and hence $(1-\eta)\dot{k}_1 = \eta\dot{k}_2$ and $\dot{k}_1 = \eta\dot{k}$ from $k_1 = \eta k$ and $k_2 = (1-\eta)k$. Here we note $f(k) = \theta\eta^{\xi-1}k^{\xi} = (1-\eta)^{\xi-1}k^{\xi}$. Then it follows that $ex_1 = f_1(k_1) - c_1 - \dot{k}_1 = f_1(k_1) - c_1$ $-\eta\dot{k} = f_1(k_1) - c_1 - \eta(f_1(k_1) + f_2(k_2) - c_1 - c_2) = (1-\eta)f_1(k_1) - \eta f_2(k_2) - (1-\eta)c_1 + \eta c_2$ $= -(1-\eta)c_1 + \eta c_2$ in view of $f_1/f_2 = \theta(k_1/k_2)^{\xi} = \theta^{1/(1-\xi)} = \eta/(1-\eta)$. Since $c_1 < c_2$ and $\eta/(1-\eta) \ge 1$ hold, $ex_1 > 0$ follows.

Lastly we show

$$F_1(t) = -\int_t^{\infty} ex_1\theta(t,\tau)d\tau \tag{47}$$

where $F_1 = F_1(t) = m_1 - k_1 > 0 (< 0)$, i.e., the net foreign asset (debt) holding by the home consumer and $\theta(t,\tau) = e^{-\int_t^{\infty} r(s)ds}$, the time discount rate. In fact, from the flow budget constraint (16) and $F_1 = m_1 - k_1$, we obtain

$$\dot{F}_1 = \dot{m}_1 - \dot{k}_1 = rm_1 + w_1 - c_1 - \dot{k}_1 = r(F_1 + k_1) + w_1 - c_1 - \dot{k}_1 = rF_1 + f_1(k_1) - c_1 - \dot{k}_1$$
$$= rF_1 + ex_1,$$

in short

$$\dot{F}_1 = rF_1 + ex_1. \tag{48}$$

From the transversality condition $\tilde{\lambda}_i m_i \to 0$ as $t \to \infty$, $i = 1,2$ (22) and $\lambda k \to 0$ as $t \to \infty (\tilde{\lambda}_1 = \tilde{\lambda}_2 = \lambda$ and hence $\lambda(m_1 + m_2) \to 0$ as $t \to \infty \Leftrightarrow \lambda k \to 0$ as $t \to \infty \Leftrightarrow$

$\lambda k_i \rightarrow 0$ as $t \rightarrow \infty$, $i = 1,2$) imply $\lambda F_i \rightarrow 0$ as $t \rightarrow \infty$, $i = 1,2$. Then from (22), we obtain

$$\tilde{\lambda}_i(t) = \tilde{\lambda}_i(0)e^{-\int_0^t r(t)d\tau} = \tilde{\lambda}_i(0)\theta(0,t), \; i = 1,2.$$

Substituting this into the transversality condition $\tilde{\lambda}_i F_i \rightarrow 0$ as $t \rightarrow \infty$, $i = 1,2$, we obtain

$$\text{NPG(No-Ponz-Game) Condition: } \lim_{t\to\infty} F_i \theta(0,t) = 0. \tag{49}$$

Next by integrating the flow budget constraint (48), we obtain

$$F_1(t) = F_1(t_1)\theta(t,t_1) + \int_{t_1}^{t} ex_1\theta(t_1,\tau)d\tau.$$

By letting $t_1 \rightarrow \infty$, and from NPG, we obtain (47).

Now we obtain

Theorem 5

(1) In the growing world economy, the more patient home country remains an exporter of good as well as debtor throughout transitional period.

(2) In the decreasing world economy, if the more impatient home country's consumption is less than that of the home country initially, then this difference remains thereafter and the home country remains an exporter of good as well as debtor throughout transitional period. If the home country's consumption is larger than that of the foreign country, then after a certain period, this difference is reserved, and the home country becomes an exporter of good as well as debtor and remains so thereafter.

We note that Theorem 5 holds even if there exists no technological superiority of the home country, i.e., even if $\theta = 1$. In this case such characteristics of trade patterns and asset-debt position arises purely from the differences in the time preference rate of consumption. Furthermore with $\theta = 1$, $ex > 0$ if and only if $c_1 < c_2$ holds, i.e., the home country is an exporter of good if and only if the home consumption is less than that of the foreign country. Furthermore the conclusion (1) of this theorem seems to be consistent with the rapid economic growth of some Asian countries (including Japan and Korea) led by export as engines of growth. In these countries saving propensities are higher reflecting lower consumption.

IV. Concluding Remarks

It is not difficult to introduce government expenditure into the model as far as it does not affect consumption nor production. Also to generalize into multi-country model would not be so difficult as far as felicity function u_i is of the same type as assumed in the last section.

Perhaps one of the most crucial assumption for the cases of trade patterns and asset-debt position is the Cobb-Douglas production function. In fact owing to this, the home countries' capital k_1 is always propotional to foreign countries' capital k_2, i.e., $k_1 = \eta(1-\eta)^{-1} k_2$.

This does not hold even if we generalize production function into C.E.S. type. One of the merits of introducing recursive type preference in the open model is that we can introduce capital accumulation into the model. In fact if we restrict to different, not endogenously determined but fixed time preference rates ρ_i, $i = 1,2$, then we have to assume away capital accumulation to let the model work as done by Ikeda and Ono (1992), for with capital accumulation, $\rho_i = f'(\bar{k})$, $i = 1,2$ must hold at the stationary state.

We have tried to send two main messages in this paper. One is to show the characteristics of the optimal consumption path, trade patterns and asset-debt positions in the globally dynamic context. Second is that the trade surplus and foreign debt of the more impatient country is the results of these countries' optimal choice. Hence if so, it does not make sense trying to let this country realize trade balance under the cause of "fair trade".

Appendix I

I. Derivation of (30)

Let $\dot{c}_i$, $\dot{\phi}_i$ and $\dot{k}$ be linearlized around the stationary state *E*; then we obtain

$$\dot{c}_1 = -a_1(\phi_1 - \overline{\phi}_1) - b_1(k - \overline{k})$$

where $-a_1 = (v_1' - \phi_1 u_1')(v_1'' - \phi_1 u_1'')^{-1} \rho_1 \cdot u_1'(v_1' - \phi_1 u_1')^{-1} = (v_1'' - \phi_1 u_1'')^{-1} \overline{r} u_1' < 0$ from $\overline{\rho}_i = \overline{r}$, $i = 1, 2$, and $\rho_{1\phi_1} = \rho_1 u_1'(v_1' - \phi_1 u_1')^{-1}$, and $b_1 = (v_1' - \phi_1 u_1')(v_1'' - \phi_1 u_1'')^{-1} f'' > 0$

observing $\rho_{1c_1} = \{(u_1 v_1'' - v_1 u_1'')(v_1' - \phi_1 u_1') - (u_1 v_1' - v_1 u_1')(v_1'' - \phi_1 u_1'')\}/(v_1 - \phi_1 u_1')^2 = 0$ from

$(u_1 v_1'' - v_1 u_1'')(v_1' - \phi_1 u_1') - (u_1 v_1' - v_1 u_1')(v_1'' - \phi_1 u_1'') = \{u_1 v_1'' - v_1 u_1'' - (v_1'' - \phi_1 u_1'')$
$(u_1 v_1' - v_1 u_1')(v_1 - \phi_1 u_1')^{-1}\}(v_1 - \phi_1 u_1') =$ (from (13) $= \{u_1 v_1'' - v_1 u_1'' - \overline{\rho}_1 (v_1'' - \phi u_1'')\}\ (v_1 - \phi_1 u_1')$
$=$(from $u_1 = \rho_1$ at *E*) $= -u_1''(v_1 - \rho_1 \cdot \phi_1)(v_1 - \phi_1 u_1') =$ (from (6), $\phi_1 = v_1 / u_1 = v_1 / \rho_1$ at $= E$) $= 0$.

Similarly

$$\dot{c}_2 = -a_2(\phi_2 - \overline{\phi}_2) - b_2(k - \overline{k})$$

where $-a_2 = (v_2' - \phi_2 u_2')(v_2'' - \phi_2 u_2'')^{-1} \rho_2 \cdot u_2'(v_2' - \phi_2 u_2')^{-1} = (v_2'' - \phi_2 u_2'')^{-1} \rho_2 u_2' < 0$, $\rho_{2\phi_2} = \rho_2 u_2'(v_2' - \phi_2 u_2')^{-1}$, $b_2 = (v_2' - \phi_2 u_2')(v_2'' - \phi_2 u_2'')^{-1} f'' > 0$ and $\rho_{2c_2} = 0$. For $\dot{\phi}_i$ and $\dot{k}$, we obtain

$$\dot{\phi}_i = -e_i(c_i - \overline{c}_i) + \overline{u}_i(\phi_i - \overline{\phi}_i)$$

and $\dot{k} = -(c_1 - \overline{c}_1) - (c_2 - \overline{c}_2) + f'(\overline{k})(k - \overline{k})$ where $e_i = v_i' - \phi_i u_i' > 0$ and $\overline{u}_i = \overline{\rho}_i =$ $\bar{f}' = \overline{r}$. Lastly for $\dot{m}_1$

$$\dot{m}_1 = -(c_1 - \overline{c}_1) + \overline{F}_1 f''(k - \overline{k}) + \overline{r}(m_1 - \overline{m}_1)$$

where $F_1 = m_1 - k_1 > 0 (\text{resp.} - F_1)$ being the home country's net foreign asset (resp. debt) holding of $F_1 > 0$ (resp. $-\overline{F}_1 > 0$). Here we note $\partial(w_1 + rm_1)/\partial k =$
$(\partial(w_1 + rm_1)/\partial k_1) \cdot dk_1/dk = \partial(f_1 - k_1 f_1' + f_1' \cdot m_1)/\partial k_1 \cdot dk_1/dk =$
$F_1 f_1'' dk_1/dk = F_1 f_1'' \cdot f''/f_1'' = F_1 f''$ from $f_1'(k_1) = f_2'(k_2) = f'(k)$. Then we obtain (30),

$$\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{k} \\ \dot{m}_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -a_1 & 0 & -b_1 & 0 \\ 0 & 0 & 0 & -a_2 & -b_2 & 0 \\ -e_1 & 0 & \bar{r} & 0 & 0 & 0 \\ 0 & -e_2 & 0 & \bar{r} & o & 0 \\ -1 & -1 & 0 & 0 & \bar{r} & 0 \\ -1 & 0 & 0 & 0 & F_1 f" & \bar{r} \end{pmatrix} \begin{pmatrix} c_1 - \bar{c}_1 \\ c_2 - \bar{c}_2 \\ \phi_1 - \bar{\phi}_1 \\ \phi_2 - \bar{\phi}_2 \\ k - \bar{k} \\ m_1 - \bar{m}_1 \end{pmatrix} \tag{30}$$

Let A be the coefficient matrix of (30). Then we observe

$$f(\omega) = |A - \omega I| = (\bar{r} - \omega)(B - \omega I) = (\bar{r} - \omega)^2 g(\omega)$$

where

$$g(\omega) = \omega^2 (\bar{r} - \omega)^2 + (a_1 e_1 + a_2 e_2 + b_1 + b_2)\omega(\bar{r} - \omega) + a_1 a_2 e_1 e_2 + e_2 a_2 b_1 + e_1 a_1 b_2 . \tag{A-1}$$

Let

$$\lambda = \omega(\bar{r} - \omega) \tag{A-2}$$

and

$$h(\lambda) = \lambda^2 + (a_1 e_1 + a_2 e_2 + b_1 + b_2)\lambda + a_1 a_2 e_1 e_2 + e_2 a_2 b_1 + e_1 a_1 b_2 . \tag{A-3}$$

Then by construction

$$h(\omega(\bar{r} - \omega)) = g(\omega) \tag{A-4}$$

holds. Next by expressing $h(\lambda)$ as

$$h(\lambda) = (\lambda + a_1 e_1 + b_1)(\lambda + a_2 e_2 + b_2) - b_1 b_2$$

and letting $\overline{B} = \max(-a_1 e_1 - b_1, -a_2 e_2 - b_2) < 0$ and $\hat{B} = \min(-a_1 e_1 - b_1, -a_2 e_2 - b_2) < 0$

we obtain for the two negative solutions of $h(\lambda) = 0$, λ and λ_2 such that

Fig. A.1

$$\lambda_2 < \hat{B} < \overline{B} < \lambda_1 < 0$$

holds. (See Fig. A. 1.)

More specifically

(1) $0 < a_1 e_1 < a_2 e_2$

$h(-a_2 e_2) < 0 < h(-a_1 e_1)$ holds.

Then it follows that

$$\lambda_2 < -a_2 e_2 < \lambda_1 < -a_1 e_1 < 0 \tag{A-5}$$

(2) $0 < a_2 e_2 < a_1 e_1$

Then

$h(-a_1 e_1) < 0 < h(-a_2 e_2)$ holds. Hence

$$\lambda_2 < -a_1 e_1 < \lambda_1 < -a_2 e_2 < 0 \tag{A-6}$$

(3) $a_1e_1 = a_2e_2$.

Then $0 > \lambda_1 = -a_1e_1 > \lambda_2 = -a_2e_2 - b_1 - b_2$ (A-7)

follows.

Devereux and Shi (1991) specified

$$v_i = -1 \text{ and } u_i(c_i) = \delta_i + \log(c_i + \alpha),\ \ i = 1, 2$$

with $\delta_1 > \delta_2 > 0$ and $\alpha > 1$. Then $a_ie_i = 1/\bar{r}$, $i = 1, 2$ follows. In short (3) corresponds to their case.

Now let ω_i, $i = 1, 2$ be the two negative solutions of $g(\omega) = 0$, i.e.,

$$\lambda_i = \omega_i(\bar{r} - \omega_i),\ \ i = 1, 2. \tag{A-8}$$

Then we observe

$$\omega_i = \frac{\bar{r} - \sqrt{\bar{r}^2 - 4\lambda_i}}{2},\ \ i = 1, 2 \tag{A-9}$$

with $\omega_2 < \omega_1 < 0$. Now we can conclude the stationary state E is locally a saddle point with two dimensional manifold of optimal path.

Now we show the local representation of c_i, ϕ_i, $i = 1, 2$ and *k* near the stationary state.

II. Local Representation of c_i, ϕ_i, $i = 1, 2$, *k* and *m*$_1$

Let $(\mu_{i1}, \mu_{i2}, \eta_{i1}, \eta_{i2}, \xi_i, \varphi_i)'$ satisfy

$$\begin{pmatrix} -\omega_i & 0 & -a_1 & 0 & -b_1 & 0 \\ 0 & -\omega_i & 0 & -a_2 & -b_2 & 0 \\ -e_1 & 0 & \bar{r} - \omega_i & 0 & 0 & 0 \\ 0 & -e_2 & 0 & \bar{r} - \omega_i & 0 & 0 \\ -1 & -1 & 0 & 0 & \bar{r} - \omega_i & 0 \\ -1 & 0 & 0 & 0 & F_1 f'' & \bar{r} - \omega_i \end{pmatrix} \begin{pmatrix} \mu_{i1} \\ \mu_{i2} \\ \eta_{i1} \\ \eta_{i2} \\ \xi_i \\ \varphi_i \end{pmatrix} = 0,\ \ i = 1, 2. \tag{A-10}$$

Then we obtain the following equations;

$$\left.\begin{aligned} c_j - \bar{c}_j &= A_1\mu_{1j}e^{\omega_1 t} + A_2\mu_{2j}e^{\omega_2 t},\ \ j = 1, 2 \\ \phi_j - \bar{\phi}_j &= A_1\eta_{1j}e^{\omega_1 t} + A_2\eta_{2j}e^{\omega_2 t},\ \ j = 1, 2 \\ k - \bar{k} &= A_1\xi_1 e^{\omega_1 t} + A_2\xi_2 e^{\omega_2 t}, \end{aligned}\right. \tag{A-11}$$

and

$$m_1 - \bar{m}_1 = A_1\varphi_1 e^{\omega_1 t} + A_2\varphi_2 e^{\omega_2 t}.$$

From (A-10), we obtain, for $i = 1, 2$

(i) $-\omega_i\mu_{i1} - \eta_{i1}a_1 - b_1\xi_i = 0$

(ii) $-\omega_i\mu_{i2} - \eta_{i2}a_2 - b_2\xi_i = 0$,

(iii) $-e_1\mu_{i1} + (\bar{r} - \omega_i)\eta_{i1} = 0$,

(iv) $-e_2\mu_{i2} + (\bar{r} - \omega_i)\eta_{i2} = 0$,

(v) $-\mu_{i1}-\mu_{i2}+(\bar{r}-\omega_i)\xi_i=0$.

and

(vi) $-\mu_{i1}+\bar{F}_1 f''(\bar{k})\cdot\xi_i+(\bar{r}-\omega_i)\varphi_i=0$, $i=1,2$

From (i) and (iii) by deleting η_{i1}, we obtain

$$-\omega_i\mu_{i1}-b_1\xi_i-a_1e_1\mu_{i1}/(\bar{r}-\omega_i)=0$$

or

$$-\mu_{i1}\frac{\omega_i(\bar{r}-\omega_i)+a_1e_1}{\bar{r}-\omega_i}=b_1\xi_i,$$

or

(vii) $$-\mu_{i1}\frac{\lambda_i+a_1e_1}{\bar{r}-\omega_i}=b_1\xi_i.$$

For (1) and (2), $\lambda_i\neq -a_1e_1$, $i=1,2$ from (A-5) and (A-6), and hence we may assume $\xi_i\neq 0$, $i=1,2$ from (vii). Hence for (1) and (2), let $\xi_1=\xi_2=1$. Then we obtain from (i) through (vii)

$$\left.\begin{aligned}
c_1-\bar{c}_1&=-A_1\frac{b_1(\bar{r}-\omega_1)}{\lambda_1+a_1e_1}e^{\omega_1 t}-A_2\frac{b_1(\bar{r}-\omega_2)}{\lambda_2+a_1e_1}e^{\omega_2 t},\\
c_2-\bar{c}_2&=-A_1\frac{b_2(\bar{r}-\omega_1)}{\lambda_1+a_2e_2}e^{\omega_1 t}-A_2\frac{b_2(\bar{r}-\omega_2)}{\lambda_2+a_2e_2}e^{\omega_2 t},\\
\phi_1-\bar{\phi}_1&=-A_1\frac{e_1b_1}{\lambda_1+a_1e_1}e^{\omega_1 t}-A_2\frac{e_1b_1}{\lambda_2+a_1e_1}e^{\omega_2 t},\\
\phi_2-\bar{\phi}_2&=-A_1\frac{e_2b_2}{\lambda_1+a_2e_2}e^{\omega_1 t}-A_2\frac{e_2b_2}{\lambda_2+a_2e_2}e^{\omega_2 t},\\
k-\bar{k}&=A_1e^{\omega_1 t}+A_2e^{\omega_2 t},\\
&\text{and}\\
m_1-\bar{m}_1&=A_1\frac{\dfrac{-b_1(r-\omega_1)}{\lambda_1+a_1e_1}-\bar{F}_1f''(\bar{k})}{\bar{r}-\omega_1}e^{\omega_1 t}+A_2\frac{\dfrac{-b_1(r-\omega_2)}{\lambda_2+a_1e_1}-\bar{F}_1f''(\bar{k})}{\bar{r}-\omega_2}e^{\omega_2 t}.
\end{aligned}\right\}\quad\text{(A-12)}$$

For (3), $\lambda_1=-a_1e_1$ and $\lambda_2=-a_2e_2-b_1-b_2$ from (A-7), implies $\xi_1=0$ from (vii). Hence let $\mu_{11}=1$ and $\xi_2=1$ for (3). Then we obtain

$$
\left.\begin{aligned}
c_1 - \bar{c}_1 &= A_1 e^{\omega_1 t} + A_2 \frac{b_1}{b_1+b_2}(\bar{r}-\omega_2)e^{\omega_2 t},\\
c_2 - \bar{c}_2 &= -A_1 e^{\omega_1 t} + A_2 \frac{b_2}{b_1+b_2}(\bar{r}-\omega_2)e^{\omega_2 t},\\
\phi_1 - \bar{\phi}_1 &= -A_1 \frac{\omega_1}{a_1} e^{\omega_1 t} + A_2 \frac{e_1 b_1}{b_1+b_2} e^{\omega_2 t},\\
\phi_2 - \bar{\phi}_2 &= A_1 \frac{\omega_1}{a_2} e^{\omega_1 t} + A_2 \frac{e_2 b_2}{b_1+b_2} e^{\omega_2 t},\\
k - \bar{k} &= A_2 e^{\omega_2 t}\\
\text{and}\quad & \\
m_1 - \bar{m}_1 &= \frac{A_1}{\bar{r}-\omega_1} e^{\omega_1 t} + A_2 \frac{\frac{b_1}{b_1+b_2}(\bar{r}-\omega_2) - F_1 f''(\bar{k})}{\bar{r}-\omega_2} e^{\omega_2 t}.
\end{aligned}\right\} \qquad \text{(A-13)}
$$

III. Determination of A_1 and A_2

Here from (A-12) and (A-13), by letting $t = 0$, we obtain

$$
k_0 - \bar{k} = \begin{cases} A_1 + A_2 & \text{for (1) and (2)} \\ A_2 & \text{for (3),} \end{cases}
$$

where k_0 is the initial value of $k = k(t)$, i.e., $k_0 = k(0)$.

$$
m_{10} - \bar{m}_1 = \begin{cases} A_1 \left\{ \dfrac{-b_1}{\lambda_1 + a_1 e_1} - \dfrac{\bar{F}_1 f''}{\bar{r}-\omega_1} \right\} + A_2 \left\{ \dfrac{-b_1}{\lambda_2 + a_1 e_1} - \dfrac{\bar{F}_1 f''}{\bar{r}-\omega_2} \right\} & \text{for(2) and (3)} \\ A_1 / (\bar{r}-\omega_1) + A_2 \left\{ \dfrac{b_1}{b_1+b_2} - \dfrac{\bar{F}_1 f''}{\bar{r}-\omega_2} \right\} & \text{for(3)} \end{cases}
$$

where m_{10} is the initial value of $m_1 = m_1(t)$, i.e., $m_{10} = m_1(0)$.

From these, A_1 and A_2 are determined as

$$
A_1 = \begin{cases} \dfrac{m_{10} - \bar{m}_1 + \{b_1/(\lambda_2 + a_1 e_1) + \bar{F}_1 f''/(\bar{r}-\omega_2)\}(k_0 - \bar{k})}{-b_1/(\lambda_1 + a_1 e_1) - \bar{F}_1 f'' \{(\bar{r}-\omega_1)^{-1} - (\bar{r}-\omega_2)^{-1}\} + b_1/(\lambda_2 + a_1 e_1)} & \text{for (1) and (2)} \\[2ex] \left\{ m_{10} - \bar{m}_1 - (k_0 - \bar{k}) \left(\dfrac{b_1}{b_1+b_2} - \dfrac{\bar{F}_1 f''}{\bar{r}-\omega_2} \right) \right\} (\bar{r}-\omega_1) & \text{for (3).} \end{cases}
$$

$$
A_2 = \begin{cases} \dfrac{-\{b_1/(\lambda_1 + a_1 e_1) + \bar{F}_1 f''/(\bar{r}-\omega_1)\}(k_0 - \bar{k}) - m_{10} + \bar{m}_{11}}{-b_1/(\lambda_1 + a_1 e_1) - \bar{F}_1 f'' \{(\bar{r}-\omega_1)^{-1} - (\bar{r}-\omega_2)^{-1}\} + b_1/(\lambda_2 + a_1 e_1)} & \text{for (1) and (2)} \\[2ex] k_0 - \bar{k} & \text{for (3)} \end{cases}
$$

.

Appendix II

Proof of Theorem 4

I Case $A_1 / A_2 < 0$

Fig. A.2.a

Fig. A.2.a shows the diagram of $f'(k)$, ρ_1 and ρ_2 for $A_1 / A_2 < 0$. Recalling $f'(\bar{k}) = \rho_1 = \rho_2$ holds at the stationary state and $\rho_1 > f'(k) > \rho_2$ holds for $k < \bar{k}$ near $\bar{k}$, we obtain $\rho_1 < f'(k) < \rho_2$ holds for $k > \bar{k}$ near $\bar{k}$ from $\bar{c}_2 < c_2$ and $\bar{c}_1 > c_1$ with $c_2'(k) < 0$ and $c_2'(k) > 0$ for k near $\bar{k}$ ($k > \bar{k}$). In Fig. A.2.a, ρ_1 curve intersects with $f'(k)$ with $\rho_1'(k') = 0$ at the intersection ($k = k'$). If this ever occurs, then the change in $c_1'(k)$ occurs at this intersection. Furthermore since c_1 curve is negatively sloped at the $k > k'$ as shown later, ρ_1 curve is positively sloped after $k > k'$, implying these two intersects only once at $k = k'$.

For ρ_2 curve to intersect with $f'(k)$ curve, $\rho_2'(k) = 0$ must hold at the intersection from $\rho_i'(k) > 0 \Leftrightarrow \phi_i'(k) > 0$ (from $\rho_i = -1/\phi_i(k)$) $\Leftrightarrow c_i'(k) > 0$ (as shown below in Fig. A.3) $\Leftrightarrow \dot{c}_i < 0$ (from $\dot{k} < 0$) $\Leftrightarrow$ $\rho_i < f'(k)$ (shown later), which is impossible since $f'(k)$ is negatively sloped. In short ρ_2 curve never intersects with $f'(k)$ curve except at $k = \bar{k}$. This shows c_2 curve is positively sloped for $k > \bar{k}$.

Fig. A.3

Fig. A.3 shows a $\phi_i'(k) > 0 \Leftrightarrow c_i'(k) > 0$, $i = 1, 2$ to hold. Recalling

$$\operatorname{sgn} \left. \frac{d\phi_i}{dk} \right|_{\dot{\phi}_i = 0} = \operatorname{sgn} c_i'(k),$$

and $\dot{\phi}_i > 0$ (resp. $<$ 0) above(resp. below) $\dot{\phi}_i = 0$ curve, we can obtain $\phi_i(k)$ curve. By construction

$$\phi_i'(k) > 0 \Leftrightarrow c_i'(k) > 0, \quad i = 1, 2.$$

From the above arguments we can obtain c_i, $i = 1, 2$ curve as drawn below;

Fig. A.4.a

Although c_1 and c_2 curve are shown to intersect at E_1, this occurs only when ρ_1 curve intersects with $f'(k)$ curve. If not, then ρ_1 is always less than $f'(k)$ for $k > \bar{k}$ and hence c_1 curve is negatively sloped for $k > \bar{k}$. Since c_2 curve is positively sloped always for $k > \bar{k}$, these two curves never meet when c_1 curve is negatively sloped for

$k > \bar{k}$. We can show these two curves meet just once at E_1, for $k > \bar{k}$ by way of contradiction supposing two curves meet at E_2 north east of E_1, as shown in Fig. A.5.

Fig. A.5

We can employ the arguments for Fig. 7b once again from (41) to (44) with interchanging the role of E_1, with E_2, and obtain contradiction.

Next we show that c_1 curve and c_2 curve never touch.

Fig. A. 6 **Fig. A.7.a**

Suppose not. Then c_1 curve and c_2 curve touch at E_1 as shown in Fig. A. 6. At E_1, $\phi_1 = \phi_2$ holds from $dc_1/dk = dc_2/dk$, (39) and $\rho_i = -1/\phi_i(k)$, $i = 1, 2$. $c_1 = c_2$ implies $u_1(c_1) > u_2(c_2)$ and hence

$$0 < d\phi_2/dk < d\phi_1/dk \quad \text{at } E_1,$$

from (6), $v_i = -1$, $i = 1, 2$ and $\dot{k} < 0$. ($c_i'(k) > 0 \Leftrightarrow \phi_i'(k) > 0$, $i = 1, 2$ and hence $\dot{\phi}_i(k) = 1 + \phi_i u_i < 0$). Let k' be slightly smaller than k^1 such that $k' = k(t')$. Then as seen from Fig. A.7.a, $\phi_1(t') < \phi_2(t') \Leftrightarrow \rho_1 < \rho_2$ holds. From A. 6, we obtain

$$0 < dc_2/dk < dc_1/dk \quad \text{at } k = k', \text{ implying}$$

$$0 > (c_2 + \alpha)(f' - \rho_2) > (c_1 + \alpha)(f' - \rho_1)$$

from (39) and $\dot{k} < 0$. Hence from $c_1 < c_2$ it follows that

$$0 > f' - \rho_2 > f' - \rho_1 \Leftrightarrow \rho_2 < \rho_1,$$

a contradiction. This shows c_1 curve and c_2 curve never touch. Hence we obtain the results of Theorem 4 for case $A_1/A_2 < 0$.

II. Case $A_1/A_2 > 0$

Next we consider case $A_1/A_2 < 0$.

Fig. A.2.b

For this case the role of ρ_1 and ρ_2 are interchanged as shown in Fig. A.2.b.

Fig. A.4.b

The slope of c_1 curve is positively sloped for $k > \bar{k}$, while slope of c_2 curve may change once at *k'*. The two curves intersect just once at E_1 if these ever do, but not more than once. We can show this by way of contradiction as drawn in Fig. A.5.

We can employ the arguments for Fig.7a.(At E_2 $\dot{c}_1 \geq \dot{c}_2 \Leftrightarrow \rho_1 \leq \rho_2 \Leftrightarrow \phi_1 \leq \phi_2$. However $u_1(c_1) > u_2(c_2)$ for $\bar{k} < k < k^2$ where $c_1 = c_2$ and $k = k^2$ at the intersection E_2 implies $\phi_1 > \phi_2$, a contradiction.)

Next we show c_1 curve and c_2 curve never touch. Suppose not. Then we obtain from Fig. A.6(Fig. A.6 holds for both $A_1 / A_2 < 0$ and $A_1 / A_2 > 0$), $\phi_1 = \phi_2$ at E_1 where $k = k_1' = k(t_1)$. However as seen from Fig. A.4.b with E_1 being touching point of the two curves, $\phi_1(t_1) > \phi_2(t_1)$ must hold since $u_1(c_1) > u_2(c_2)$ for $\bar{k} < k \le k^1$, a contradiction.

Hence c_1 curve and c_2 curve never touch. Hence we obtain the results of Theorem 4 for $A_1 / A_2 > 0$.

Figures

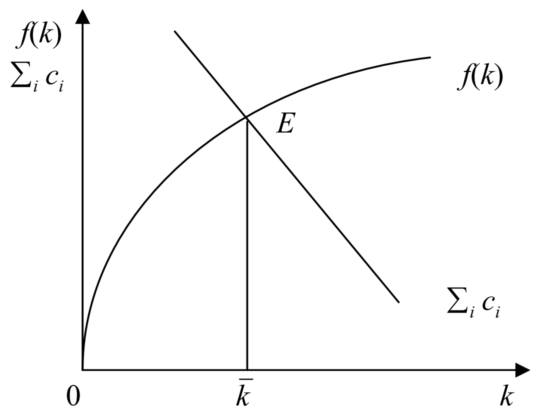

Fig. 1 Existence Uniqueness of Stationary State

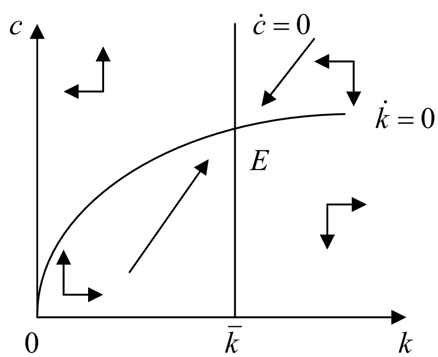

Fig.2a $\xi'(k) \le 0$

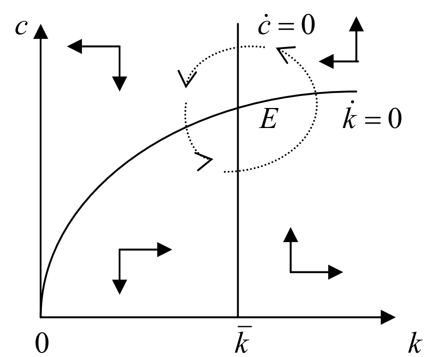

Fig.2b $\xi'(k) > 0$

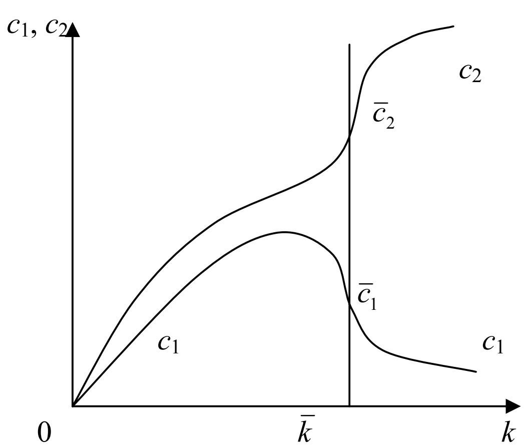

Fig. 3a $(A_1 / A_2 < 0)$

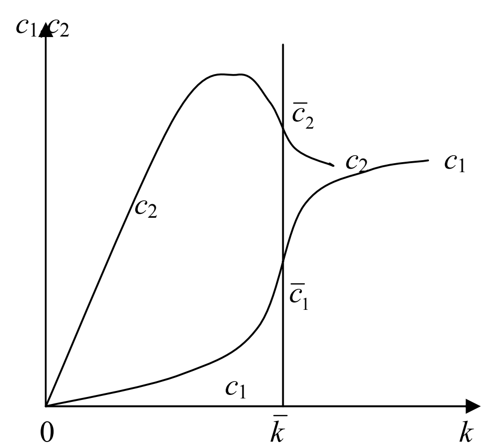

Fig. 3b $(A_1 / A_2 > 0)$

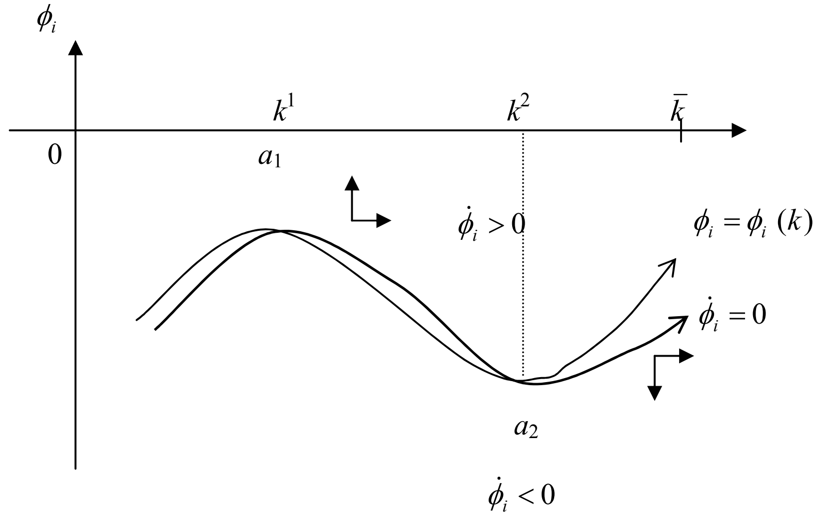

Fig. 4

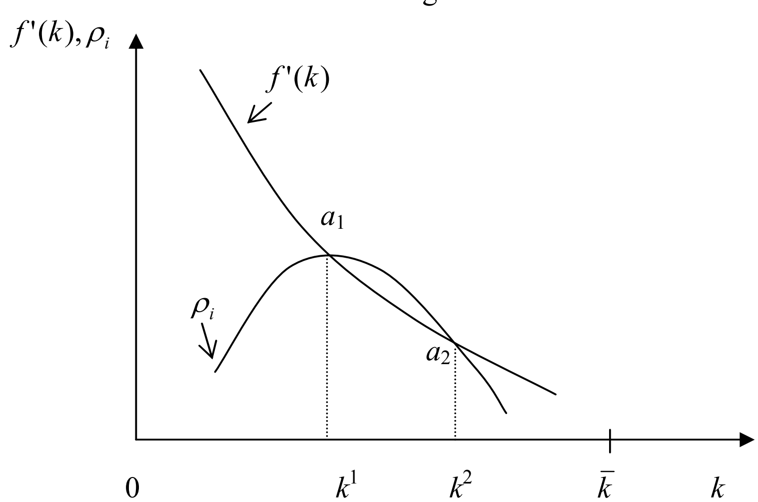

Fig. 5

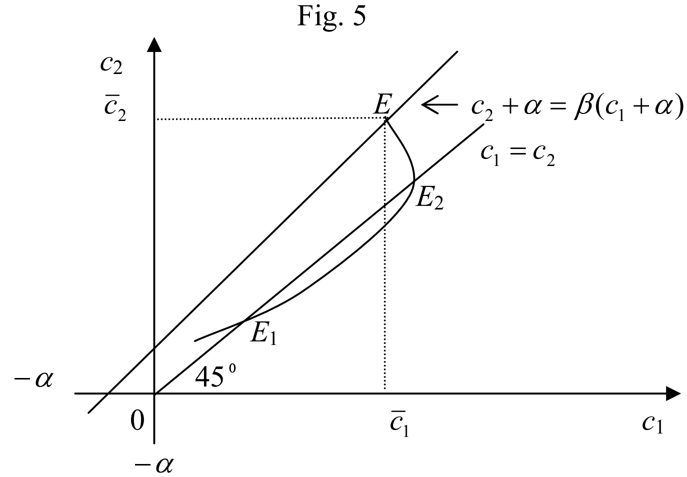

Fig. 6

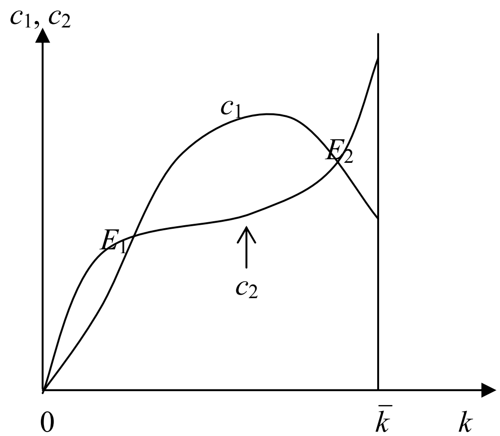

Fig. 7a

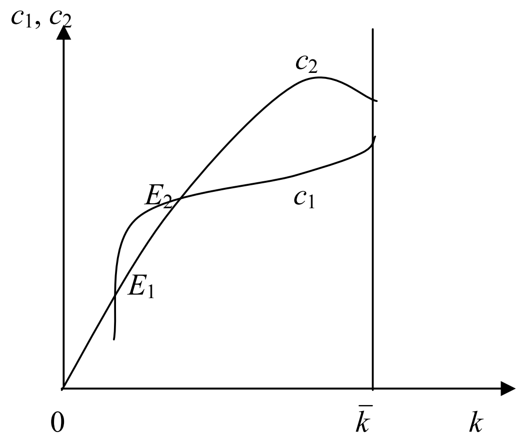

Fig. 7b

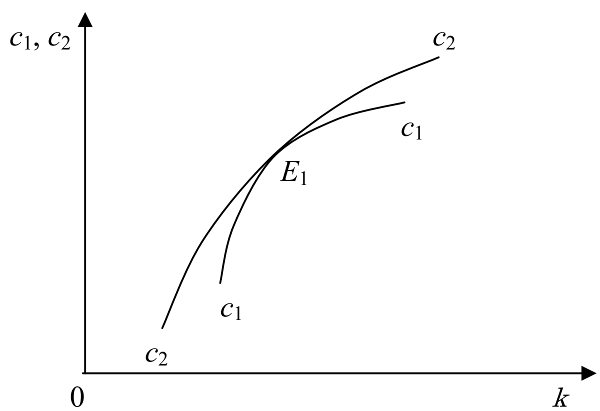

Fig. 8

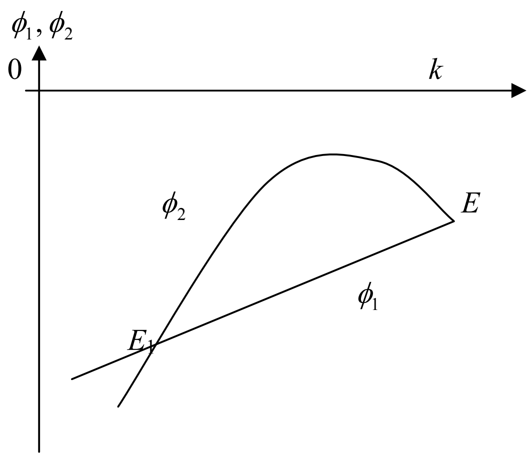

Fig. 9

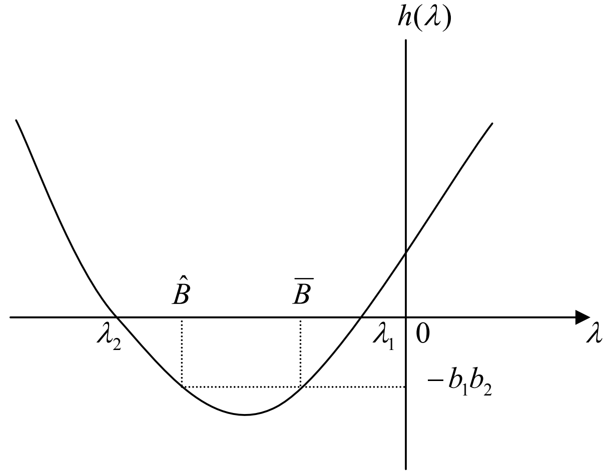

Fig. A.1

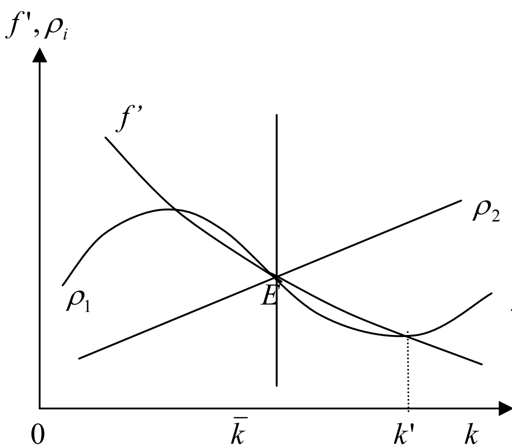

Fig. A.2.a

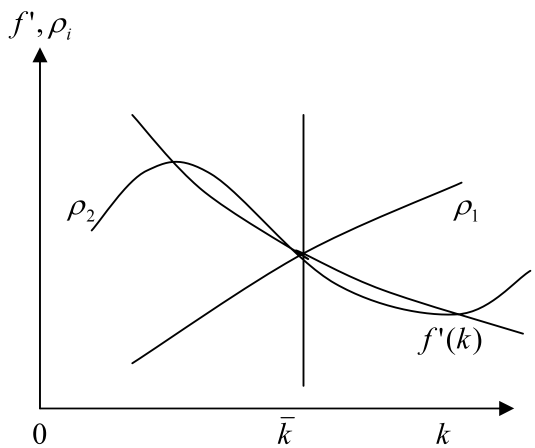

Fig. A.2.b

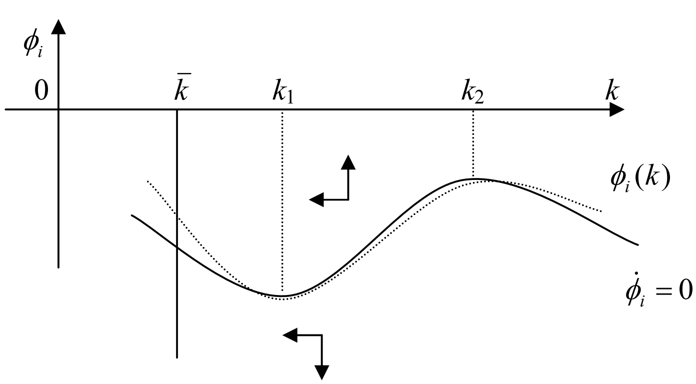

Fig. A.3

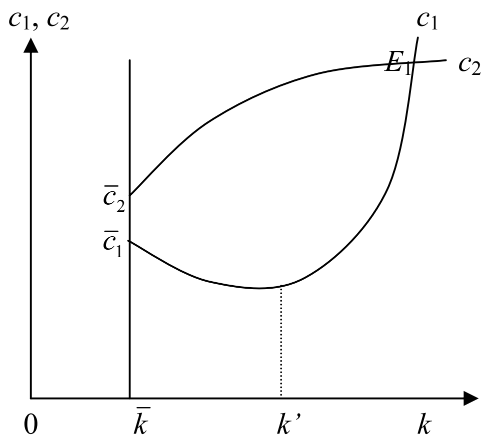

Fig. A.4a

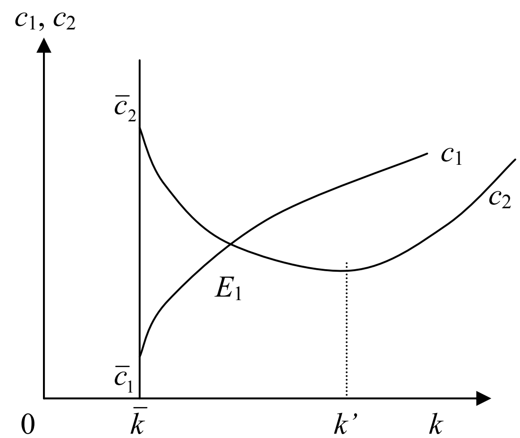

Fig. A.4b

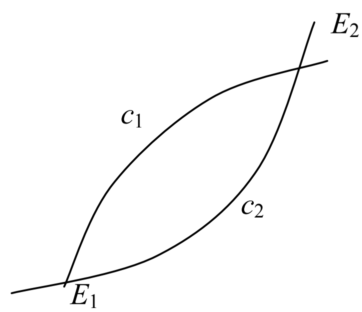

Fig. A.5

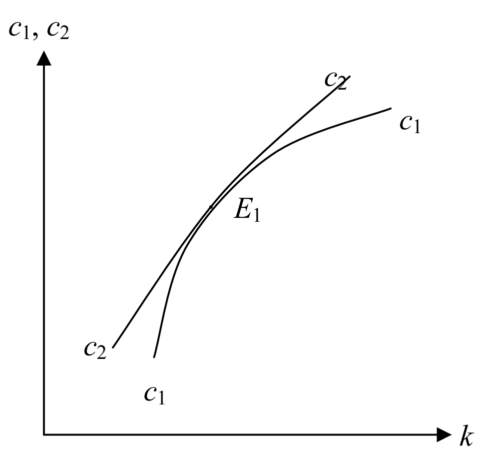

Fig. A.6

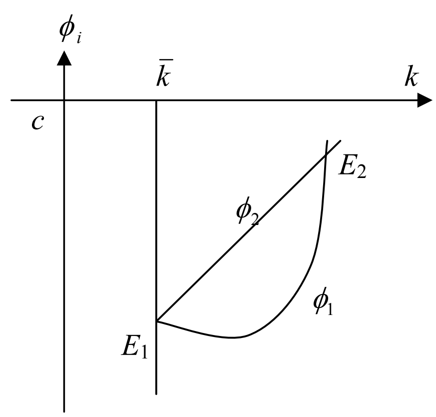

Fig. A.7a

Notes

1. Devereux and Shi (1991) employed the technique developed by Epstein (1987a), showing that for the upper-left 2x2 matrix of $A(\bar{r}I - A) = \begin{pmatrix} -a_1e_1b_1 & -b_1 \\ -b_2 & -a_2e_2-b_2 \end{pmatrix} = M$, $h(\lambda) = |M - \lambda I| = 0$ has two negative solutions λ_1 and λ_2, and (A-4) holds when $v_i = -1$, and $u_i(c_i) = \delta_i + \log(c_i + \alpha)$, $i = 1, 2$, with $\delta_1 > \delta_2$ and $\alpha > 1$ are assumed.
2. Let $F(X,t) \in R^n$ be continuously differentiable in $X \in R^n$ and $t \in R$ and satisfy a Lipschitz condition. Then the system of ordinary differential equations $dX/dt = \dot{X} = F(X,t)$ has a unique solution $X = X(X_0, t)$ continuously differentiable in X_0 and *t* where X_0 is the initial value of *X*.
3. By employing the above theorems again, we observe that the solution path c_1, c_2, ϕ_1, ϕ_2 and *k* such that $c_i = \tilde{c}_i(t,\ Y_0)$, $\phi_i = \tilde{\phi}_i(t,\ Y_0)$, $i = 1,2$ and $k = \tilde{k}(t,\ Y_0)$ are continuously differentiable in $(t,\ Y_0)$ and especially for $\dot{k} = f(k) - (c_1 + c_2) = 0$. In fact let t_j, k_j and c_{ij}, $i = 1,2$ and $j = 1,2, \ldots$ be the values of *t*, *k* and c_i, $i = 1,2$ respectively such that $\dot{k} = f(k) - (c_1 + c_2) = 0$. (It is immediate to show $\dot{k} = f(k) - (c_1 + c_2) = 0$ at most for countabilly many distinct points of (k, c_1, c_2)) Then $\tilde{c}_i$, $\tilde{\phi}_i$ and $\tilde{k}$ are continuously differentiable at t_j, $j = 1,2 \ldots$ with $t_i < +\infty$. This implies $c_i(k, Y_0)$ and $\phi_i(k, Y_0)$, are continuously differentiable at $k_j (\neq \bar{k})$, $j = 1,2 \ldots$ Since $c_i(k, Y_0)$ and $\phi_i(k, Y_0)$, $i = 1,2$ are continuous in *k* at $k = \bar{k}$ also from (33-1) through (34-6), we observe these are continuously differentiable in $k > 0$ except $k = \bar{k}$ and continuous at $k = \bar{k}$.
4. For this, see. e.g. Hsu and Meyer (1968), Section 5.8.

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